

SOME GENERALIZATIONS OF SECOND SUBMODULES

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ABSTRACT. In this paper, we will introduce two generalizations of second submodules of a module over a commutative ring and explore some basic properties of these classes of modules

1. INTRODUCTION

Throughout this paper, R will denote a commutative ring with identity and " \subset " will denote the strict inclusion. Further, \mathbb{Z} will denote the ring of integers.

Let M be an R -module. A proper submodule P of M is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [14]. A non-zero submodule S of M is said to be *second* if for each $a \in R$, the homomorphism $S \xrightarrow{a} S$ is either surjective or zero [19]. In this case $\text{Ann}_R(S)$ is a prime ideal of R .

Badawi gave a generalization of prime ideals in [9] and said such ideals 2-absorbing ideals. A proper ideal I of R is a *2-absorbing ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. He proved that I is a 2-absorbing ideal of R if and only if whenever I_1, I_2 , and I_3 are ideals of R with $I_1 I_2 I_3 \subseteq I$, then $I_1 I_2 \subseteq I$ or $I_1 I_3 \subseteq I$ or $I_2 I_3 \subseteq I$. Yousefian Darani and Soheilnia in [12] extended 2-absorbing ideals to 2-absorbing submodules. A proper submodule N of M is called a *2-absorbing submodule* of M if whenever $abm \in N$ for some $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$. Several authors investigated properties of 2-absorbing submodules, for example see [12, 17, 18].

A submodule N of an R -module M is called *strongly 2-absorbing* if $IJL \subseteq N$ for some ideals I, J of R and a submodule L of M , then $IL \subseteq N$ or $JL \subseteq N$ or $IJ \in (N :_R M)$ [13].

The purpose of this paper is to introduce the dual notions of 2-absorbing and strongly 2-absorbing submodules and obtain some related results. Also, as we can see in Corollary 3.19, these are two generalizations of second submodules. In [18, 2.3], the authors show that N is a 2-absorbing submodule of an R -module M if and only if N is a strongly 2-absorbing submodule of M . The Example 3.2 shows that the dual of this fact is not true in general.

2. 2-ABSORBING SECOND SUBMODULES

Let M be an R -module. A proper submodule N of M is said to be *completely irreducible* if $N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of submodules of M , implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M [15].

We frequently use the following basic fact without further comment.

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Remark 2.1. Let N and K be two submodules of an R -module M . To prove $N \subseteq K$, it is enough to show that if L is a completely irreducible submodule of M such that $K \subseteq L$, then $N \subseteq L$.

Definition 2.2. Let N be a non-zero submodule of an R -module M . We say that N is a *2-absorbing second submodule* of M if whenever $a, b \in R$, L is a completely irreducible submodule of M , and $abN \subseteq L$, then $aN \subseteq L$ or $bN \subseteq L$ or $ab \in \text{Ann}_R(N)$. This can be regarded as a dual notion of the 2-absorbing submodule.

A non-zero R -module M is said to be *secondary* if for each $a \in R$ the endomorphism of M given by multiplication by a is either surjective or nilpotent [16].

Theorem 2.3. Let M be an R -module. Then we have the following.

- (a) If either N is a second submodule of M or N is a sum of two second submodules of M , then N is 2-absorbing second.
- (b) If N is a secondary submodule of M and $R/\text{Ann}_R(N)$ has no non-zero nilpotent element, then N is 2-absorbing second.

Proof. (a) The first assertion is clear. To see the second assertion, let N_1 and N_2 be two second submodules of M . We show that $N_1 + N_2$ is a 2-absorbing second submodule of M . Assume that $a, b \in R$, L is a completely irreducible submodule of M , and $ab(N_1 + N_2) \subseteq L$. Since N_1 is second, $abN_1 = 0$ or $N_1 \subseteq L$ by [3, 2.10]. Similarly, $abN_2 = 0$ or $N_2 \subseteq L$. If $abN_1 = 0 = abN_2$ (resp. $N_1 \subseteq L$ and $N_2 \subseteq L$), then we are done. Now let $abN_1 = 0$ and $N_2 \subseteq L$. Then $aN_1 = 0$ or $bN_1 = 0$ because $\text{Ann}_R(N_1)$ is a prime ideal of R . If $aN_1 = 0$, then $a(N_1 + N_2) \subseteq aN_1 + N_2 \subseteq N_2 \subseteq L$. Similarly, if $bN_1 = 0$, we get $b(N_1 + N_2) \subseteq L$ as desired.

(b) Let $a, b \in R$, L be a completely irreducible submodule of M , and $abN \subseteq L$. Then if $aN \subseteq L$ or $bN \subseteq L$, we are done. Let $aN \not\subseteq L$ and $bN \not\subseteq L$. Then $a, b \in \sqrt{\text{Ann}_R(N)}$. Thus, $(ab)^s \in \text{Ann}_R(N)$ for some positive integer s . Therefore, $ab \in \text{Ann}_R(N)$ because $R/\text{Ann}_R(N)$ has no non-zero nilpotent element. \square

Lemma 2.4. Let I be an ideal of R and N be a 2-absorbing second submodule of M . If $a \in R$, L is a completely irreducible submodule of M , and $IaN \subseteq L$, then $aN \subseteq L$ or $IN \subseteq L$ or $Ia \in \text{Ann}_R(N)$.

Proof. Let $aN \not\subseteq L$ and $Ia \notin \text{Ann}_R(N)$. Then there exists $b \in I$ such that $abN \neq 0$. Now as N is a 2-absorbing second submodule of M , $baN \subseteq L$ implies that $bN \subseteq L$. We show that $IN \subseteq L$. To see this, let c be an arbitrary element of I . Then $(b+c)aN \subseteq L$. Hence, either $(b+c)N \subseteq L$ or $(b+c)a \in \text{Ann}_R(N)$. If $(b+c)N \subseteq L$, then since $bN \subseteq L$ we have $cN \subseteq L$. If $(b+c)a \in \text{Ann}_R(N)$, then $ca \notin \text{Ann}_R(N)$, but $caN \subseteq L$. Thus $cN \subseteq L$. Hence, we conclude that $IN \subseteq L$. \square

Lemma 2.5. Let I and J be two ideals of R and N be a 2-absorbing second submodule of M . If L is a completely irreducible submodule of M and $IJN \subseteq L$, then $IN \subseteq L$ or $JN \subseteq L$ or $IJ \subseteq \text{Ann}_R(N)$.

Proof. Let $IN \not\subseteq L$ and $JN \not\subseteq L$. We show that $IJ \subseteq \text{Ann}_R(N)$. Assume that $c \in I$ and $d \in J$. By assumption there exists $a \in I$ such that $aN \not\subseteq L$ but $aJN \subseteq L$. Now Lemma 2.4 shows that $aJ \subseteq \text{Ann}_R(N)$ and so $(I \setminus (L :_R N))J \subseteq \text{Ann}_R(N)$. Similarly there exists $b \in (J \setminus (L :_R N))$ such that $Ib \subseteq \text{Ann}_R(N)$ and also $I(J \setminus (L :_R N)) \subseteq \text{Ann}_R(N)$. Thus we have $ab \in \text{Ann}_R(N)$, $ad \in \text{Ann}_R(N)$ and

$cb \in \text{Ann}_R(N)$. As $a+c \in I$ and $b+d \in J$, we have $(a+c)(b+d)N \subseteq L$. Therefore, $(a+c)N \subseteq L$ or $(b+d)N \subseteq L$ or $(a+c)(b+d) \in \text{Ann}_R(N)$. If $(a+c)N \subseteq L$, then $cN \not\subseteq L$. Hence $c \in I \setminus (L :_R N)$ which implies that $cd \in \text{Ann}_R(N)$. Similarly if $(b+d)N \subseteq L$, we can deduce that $cd \in \text{Ann}_R(N)$. Finally if $(a+c)(b+d) \in \text{Ann}_R(N)$, then $ab + ad + cb + cd \in \text{Ann}_R(N)$ so that $cd \in \text{Ann}_R(N)$. Therefore, $IJ \subseteq \text{Ann}_R(N)$. \square

Corollary 2.6. Let M be an R -module and N be a 2-absorbing second submodule of M . Then IN is a 2-absorbing second submodules of M for all ideals I of R with $I \not\subseteq \text{Ann}_R(N)$.

Proof. Let I be an ideal of R with $I \not\subseteq \text{Ann}_R(N)$, $a, b \in R$, L be a completely irreducible submodule of M , and $abIN \subseteq L$. Then $aN \subseteq L$ or $bIN \subseteq L$ or $abIN = 0$ by Lemma 2.4. If $bIN \subseteq L$ or $abIN = 0$, then we are done. If $aN \subseteq L$, then $aIN \subseteq aN$ implies that $aIN \subseteq L$, as needed. \square

An R -module M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$ [10].

Corollary 2.7. Let M be a multiplication 2-absorbing second R -module. Then every non-zero submodule of M is a 2-absorbing second submodule of M .

Proof. This follows from Corollary 2.6. \square

The following example shows that the condition “ M is a multiplication module” in Corollary 2.7 can not be omitted.

Example 2.8. For any prime integer p , let $M = \mathbb{Z}_{p^\infty}$ and $N = \langle 1/p^3 + \mathbb{Z} \rangle$. Then clearly, M is a 2-absorbing second \mathbb{Z} -module but $p^2 \langle 1/p^3 + \mathbb{Z} \rangle \subseteq \langle 1/p + \mathbb{Z} \rangle$ implies that N is not a 2-absorbing second submodule of M .

We recall that an R -module M is said to be a *cocyclic module* if $\text{Soc}_R(M)$ is a large and simple submodule of M [21]. (Here $\text{Soc}_R(M)$ denotes the sum of all minimal submodules of M .) A submodule L of M is a completely irreducible submodule of M if and only if M/L is a cocyclic R -module [15].

Proposition 2.9. Let N be a 2-absorbing second submodule of an R -module M . Then we have the following.

- (a) If L is a completely irreducible submodule of M such that $N \not\subseteq L$, then $(L :_R N)$ is a 2-absorbing ideal of R .
- (b) If M is a cocyclic module, then $\text{Ann}_R(N)$ is a 2-absorbing ideal of R .
- (c) If $a \in R$, then $a^n N = a^{n+1} N$, for all $n \geq 2$.
- (d) If $\text{Ann}_R(N)$ is a prime ideal of R , then $(L :_R N)$ is a prime ideal of R for all completely irreducible submodules L of M such that $N \not\subseteq L$.

Proof. (a) Since $N \not\subseteq L$, we have $(L :_R N) \neq R$. Let $a, b, c \in R$ and $abc \in (L :_R N)$. Then $abN \subseteq (L :_M c)$. Thus $aN \subseteq (L :_M c)$ or $bN \subseteq (L :_M c)$ or $abN = 0$ because by [8, 2.1], $(L :_M c)$ is a completely irreducible submodule of M . Therefore, $ac \in (L :_R N)$ or $bc \in (L :_R N)$ or $ab \in (L :_R N)$.

(b) Since M is cocyclic, the zero submodule of M is a completely irreducible submodule of M . Thus the result follows from part (a).

(c) It is enough to show that $a^2 N = a^3 N$. It is clear that $a^3 N \subseteq a^2 N$. Let L be a completely irreducible submodule of M such that $a^3 N \subseteq L$. Then $a^2 N \subseteq (L :_R a)$. Since N is 2-absorbing second submodule and $(L :_R a)$ is a completely irreducible

submodule of M by [8, 2.1], $aN \subseteq (L :_R a)$ or $a^2N = 0$. Therefore, $a^2N \subseteq L$. This implies that $a^2N \subseteq a^3N$.

(d) Let $a, b \in R$, L be a completely irreducible submodule of M such that $N \not\subseteq L$, and $ab \in (L :_R N)$. Then $aN \subseteq L$ or $bN \subseteq L$ or $abN = 0$. If $abN = 0$, then by assumption, $aN = 0$ or $bN = 0$. Thus in any cases we get that, $aN \subseteq L$ or $bN \subseteq L$. \square

Theorem 2.10. *Let N be a 2-absorbing second submodule of an R -module M . Then we have the following.*

- (a) *If $\sqrt{\text{Ann}_R(N)} = P$ for some prime ideal P of R and L is a completely irreducible submodule of M such that $N \not\subseteq L$, then $\sqrt{(L :_R N)}$ is a prime ideal of R containing P .*
- (b) *If $\sqrt{\text{Ann}_R(N)} = P \cap Q$ for some prime ideals P and Q of R , L is a completely irreducible submodule of M such that $N \not\subseteq L$, and $P \subseteq \sqrt{(L :_R N)}$, then $\sqrt{(L :_R N)}$ is a prime ideal of R .*

Proof. (a) Assume that $a, b \in R$ and $ab \in \sqrt{(L :_R N)}$. Then there is a positive integer t such that $a^t b^t N \subseteq L$. By hypotheses, N is a 2-absorbing second submodule of M , thus $a^t N \subseteq L$ or $b^t N \subseteq L$ or $a^t b^t \in \text{Ann}_R(N)$. If either $a^t N \subseteq L$ or $b^t N \subseteq L$, we are done. So assume that $a^t b^t \in \text{Ann}_R(N)$. Then $ab \in \sqrt{\text{Ann}_R(N)} = P$ and so $a \in P$ or $b \in P$. It is clear that $P = \sqrt{\text{Ann}_R(N)} \subseteq \sqrt{(L :_R N)}$. Therefore, $a \in \sqrt{(L :_R N)}$ or $b \in \sqrt{(L :_R N)}$.

(b) The proof is similar to that of part (a). \square

Proposition 2.11. *Let M be an R -module and let $\{K_i\}_{i \in I}$ be a chain of 2-absorbing second submodules of M . Then $\cup_{i \in I} K_i$ is a 2-absorbing second submodule of M .*

Proof. Let $a, b \in R$, L be a completely irreducible submodule of M , and $ab(\cup_{i \in I} K_i) \subseteq L$. Assume that $a(\cup_{i \in I} K_i) \not\subseteq L$ and $b(\cup_{i \in I} K_i) \not\subseteq L$. Then there are $m, n \in I$, where $aK_n \not\subseteq L$ and $bK_m \not\subseteq L$. Hence, for every $K_n \subseteq K_s$ and $K_m \subseteq K_d$ we have $aK_s \not\subseteq L$ and $bK_d \not\subseteq L$. Therefore, for each submodule K_h such that $K_n \subseteq K_h$ and $K_m \subseteq K_h$ we have $abK_h = 0$. Hence $ab(\cup_{i \in I} K_i) = 0$, as needed. \square

Definition 2.12. We say that a 2-absorbing second submodule N of an R -module M is a *maximal 2-absorbing second submodule* of a submodule K of M , if $N \subseteq K$ and there does not exist a 2-absorbing second submodule H of M such that $N \subset H \subset K$.

Lemma 2.13. *Let M be an R -module. Then every 2-absorbing second submodule of M is contained in a maximal 2-absorbing second submodule of M .*

Proof. This is proved easily by using Zorn's Lemma and Proposition 2.11. \square

Theorem 2.14. *Every Artinian R -module has only a finite number of maximal 2-absorbing second submodules.*

Proof. Suppose that there exists a non-zero submodule N of M such that it has an infinite number of maximal 2-absorbing second submodules. Let S be a submodule of M chosen minimal such that S has an infinite number of maximal 2-absorbing second submodules. Then S is not 2-absorbing second submodule. Thus there exist $a, b \in R$ and a completely irreducible submodule L of M such that $abS \subseteq L$ but

$aS \not\subseteq L$, $bS \not\subseteq L$, and $abS \neq 0$. Let V be a maximal 2-absorbing second submodule of M contained in S . Then $aV \subseteq L$ or $bV \subseteq L$ or $abV = 0$. Thus $V \subseteq (L :_M a)$ or $V \subseteq (L :_M b)$ or $V \subseteq (0 :_M ab)$. Therefore, $V \subseteq (L :_S a)$ or $V \subseteq (L :_S b)$ or $V \subseteq (0 :_S ab)$. By the choice of S , the modules $(L :_S a)$, $(L :_S b)$, and $(0 :_S ab)$ have only finitely many maximal 2-absorbing second submodules. Therefore, there is only a finite number of possibilities for the module S which is a contradiction. \square

3. STRONGLY 2-ABSORBING SECOND SUBMODULES

Definition 3.1. Let N be a non-zero submodule of an R -module M . We say that N is a *strongly 2-absorbing second submodule* of M if whenever $a, b \in R$, L_1, L_2 are completely irreducible submodules of M , and $abN \subseteq L_1 \cap L_2$, then $aN \subseteq L_1 \cap L_2$ or $bN \subseteq L_1 \cap L_2$ or $ab \in \text{Ann}_R(N)$. This can be regarded as a dual notion of the strongly 2-absorbing submodule.

Example 3.2. Clearly every strongly 2-absorbing second submodule is a 2-absorbing second submodule. But the converse is not true in general. For example, consider \mathbb{Z} as a \mathbb{Z} -module. Then $2\mathbb{Z}$ is a 2-absorbing second submodule of \mathbb{Z} but it is not a strongly 2-absorbing second submodule of \mathbb{Z} .

Theorem 3.3. Let N be a submodule of an R -module M . The following statements are equivalent:

- (a) N is a strongly 2-absorbing second submodule of M ;
- (b) If $N \neq 0$, $IJN \subseteq K$ for some ideals I, J of R and a submodule K of M , then $IN \subseteq K$ or $JN \subseteq K$ or $IJ \in \text{Ann}_R(N)$;
- (c) $N \neq 0$ and for each $a, b \in R$, we have $abN = aN$ or $abN = bN$ or $abN = 0$.

Proof. (a) \Rightarrow (b). Assume that $IJN \subseteq K$ for some ideals I, J of R , a submodule K of M , and $IJ \not\subseteq \text{Ann}_R(N)$. Then by Lemma 2.5, for all completely irreducible submodules L of M with $K \subseteq L$ either $IN \subseteq L$ or $JN \subseteq L$. If $IN \subseteq L$ (resp. $JN \subseteq L$) for all completely irreducible submodules L of M with $K \subseteq L$, we are done. Now suppose that L_1 and L_2 are two completely irreducible submodules of M with $K \subseteq L_1$, $K \subseteq L_2$, $IN \not\subseteq L_1$, and $JN \not\subseteq L_2$. Then $IN \subseteq L_2$ and $JN \subseteq L_1$. Since $IJN \subseteq L_1 \cap L_2$, we have either $IN \subseteq L_1 \cap L_2$ or $JN \subseteq L_1 \cap L_2$. As $IN \subseteq L_1 \cap L_2$, we have $IN \subseteq L_1$ which is a contradiction. Similarly from $JN \subseteq L_1 \cap L_2$ we get a contradiction.

(b) \Rightarrow (a). This is clear.

(a) \Rightarrow (c). By part (a), $N \neq 0$. Let $a, b \in R$. Then $abN \subseteq abN$ implies that $aN \subseteq abN$ or $bN \subseteq abN$ or $abN = 0$. Thus $abN = aN$ or $abN = bN$ or $abN = 0$.

(c) \Rightarrow (a). This is clear. \square

Lemma 3.4. Let M be an R -module, $N \subset K$ be two submodules of M , and K be a strongly 2-absorbing second submodule of M . Then K/N is a strongly 2-absorbing second submodule of M/N .

Proof. This is straightforward. \square

Proposition 3.5. Let N be a strongly 2-absorbing second submodule of an R -module M . Then we have the following.

- (a) $\text{Ann}_R(N)$ is a 2-absorbing ideal of R .
- (b) If K is a submodule of M such that $N \not\subseteq K$, then $(K :_R N)$ is a 2-absorbing ideal of R .

- (c) If I is an ideal of R , then $I^n N = I^{n+1} N$, for all $n \geq 2$.
- (d) If $(L_1 \cap L_2 :_R N)$ is a prime ideal of R for all completely irreducible submodules L_1 and L_2 of M such that $N \not\subseteq L_1 \cap L_2$, then $\text{Ann}_R(N)$ is a prime ideal of R .

Proof. (a) Let $a, b, c \in R$ and $abc \in \text{Ann}_R(N)$. Then $abN \subseteq abN$ implies that $aN \subseteq abN$ or $bN \subseteq abN$ or $abN = 0$ by Theorem 3.3. If $abN = 0$, then we are done. If $aN \subseteq abN$, then $caN \subseteq cabN = 0$. In other case, we do the same.

(b) Let $a, b, c \in R$ and $abc \in (K :_R N)$. Then $acN \subseteq K$ or $bcN \subseteq K$ or $abcN = 0$. If $acN \subseteq K$ or $bcN \subseteq K$, then we are done. If $abcN = 0$, then the result follows from part (a).

(c) It is enough to show that $I^2 N = I^3 N$. It is clear that $I^3 N \subseteq I^2 N$. Since N is strongly 2-absorbing second submodule, $I^3 N \subseteq I^3 N$ implies that $I^2 N \subseteq I^3 N$ or $IN \subseteq I^3 N$ or $I^3 N = 0$ by Theorem 3.3. If $I^2 N \subseteq I^3 N$ or $IN \subseteq I^3 N$, then we are done. If $I^3 N = 0$, then the result follows from part (a).

(d) Suppose that $a, b \in R$ and $abN = 0$. Assume contrary that $aN \neq 0$ and $bN \neq 0$. Then there exist completely irreducible submodules L_1 and L_2 of M such that $aN \not\subseteq L_1$ and $bN \not\subseteq L_2$. Now since $(L_1 \cap L_2 :_R N)$ is a prime ideal of R , $0 = abN \subseteq L_1 \cap L_2$ implies that $bN \subseteq L_1 \cap L_2$ or $aN \subseteq L_1 \cap L_2$. In any cases, we have a contradiction. \square

Remark 3.6. ([9, Theorem 2.4]). If I is a 2-absorbing ideal of R , then one of the following statements must hold:

- (a) $\sqrt{I} = P$ is a prime ideal of R such that $P^2 \subseteq I$;
- (b) $\sqrt{I} = P \cap Q$, $PQ \subseteq I$, and $\sqrt{I}^2 \subseteq I$ where P and Q are the only distinct prime ideals of R that are minimal over I .

Theorem 3.7. *If N is a strongly 2-absorbing second submodule of M and $N \not\subseteq K$, then either $(K :_R N)$ is a prime ideal of R or there exists an element $a \in R$ such that $(K :_R aN)$ is a prime ideal of R .*

Proof. By Proposition 3.5 and Remark 3.6, we have one of the following two cases.

- (a) Let $\sqrt{\text{Ann}_R(N)} = P$, where P is a prime ideal of R . We show that $(K :_R N)$ is a prime ideal of R when $P \subseteq (K :_R N)$. Assume that $a, b \in R$ and $ab \in (K :_R N)$. Hence $aN \subseteq K$ or $bN \subseteq K$ or $ab \in \text{Ann}_R(N)$. If either $aN \subseteq K$ or $bN \subseteq K$, we are done. Now assume that $ab \in \text{Ann}_R(N)$. Then $ab \in P$ and so $a \in P$ or $b \in P$. Thus, $a \in (K :_R N)$ or $b \in (K :_R N)$ and the assertion follows. If $P \not\subseteq (K :_R N)$, then there exists $a \in P$ such that $aN \not\subseteq K$. By Remark 3.6, $P^2 \subseteq \text{Ann}_R(N) \subseteq (K :_R N)$, thus $P \subseteq (K :_R aN)$. Now a similar argument shows that $(K :_R aN)$ is a prime ideal of R .
- (b) Let $\sqrt{\text{Ann}_R(N)} = P \cap Q$, where P and Q are distinct prime ideals of R . If $P \subseteq (K :_R N)$, then the result follows by a similar proof to that of part (a). Assume that $P \not\subseteq (K :_R N)$. Then there exists $a \in P$ such that $aN \not\subseteq K$. By Remark 3.6, we have $PQ \subseteq \text{Ann}_R(N) \subseteq (K :_R N)$. Thus, $Q \subseteq (K :_R aN)$ and the result follows by a similar proof to that of part (a).

\square

Let M be an R -module. A prime ideal P of R is said to be a *coassociated prime* of M if there exists a cocyclic homomorphic image T of M such that $P = \text{Ann}_R(T)$. The set of all coassociated prime ideals of M is denoted by $\text{Coass}_R(M)$ [20].

Theorem 3.8. *Let N be a strongly 2-absorbing second submodule of an R -module M . Then we have the following.*

- (a) *If $\sqrt{\text{Ann}_R(N)} = P$ for some prime ideal P of R , L_1 and L_2 are completely irreducible submodules of M such that $N \not\subseteq L_1$, and $N \not\subseteq L_2$, then either $\sqrt{(L_1 :_R N)} \subseteq \sqrt{(L_2 :_R N)}$ or $\sqrt{(L_2 :_R N)} \subseteq \sqrt{(L_1 :_R N)}$. Hence, $\text{Coass}_R(N)$ is a totally ordered set.*
- (b) *If $\sqrt{\text{Ann}_R(N)} = P \cap Q$ for some prime ideals P and Q of R , L_1 and L_2 are completely irreducible submodules of M such that $N \not\subseteq L_1$ and $N \not\subseteq L_2$, and $P \subseteq \sqrt{(L_1 :_R N)} \cap \sqrt{(L_2 :_R N)}$, then either $\sqrt{(L_1 :_R N)} \subseteq \sqrt{(L_2 :_R N)}$ or $\sqrt{(L_2 :_R N)} \subseteq \sqrt{(L_1 :_R N)}$. Hence, $\text{Coass}_R(N)$ is the union of two totally ordered sets.*

Proof. (a) Assume that $\sqrt{(L_1 :_R N)} \not\subseteq \sqrt{(L_2 :_R N)}$. We show that $\sqrt{(L_2 :_R N)} \subseteq \sqrt{(L_1 :_R N)}$. Suppose that $a \in \sqrt{(L_1 :_R N)}$ and $b \in \sqrt{(L_2 :_R N)}$. Then there exists a positive integer s such that $a^s N \subseteq L_1$, $b^s N \subseteq L_2$, and $b^s N \not\subseteq L_1$. If $a^s N \subseteq L_1 \cap L_2$, then $a^s N \subseteq L_2$ and so $a \in \sqrt{(L_2 :_R N)}$. Now assume that $a^s N \not\subseteq L_1 \cap L_2$. Then $a^s b^s \in \text{Ann}_R(N)$ because $a^s b^s N \subseteq L_1 \cap L_2$, $a^s N \not\subseteq L_1 \cap L_2$, and $b^s N \not\subseteq L_1 \cap L_2$. Thus, $ab \in P$. If $b \in P$, then $b^s N \subseteq L_1$ which is a contradiction. Hence $a \in P$ and so $a \in \sqrt{(L_2 :_R N)}$. Let $P, Q \in \text{Coass}_R(N)$. Then there exist completely irreducible submodules L_1 and L_2 of M such that $P = (L_1 :_R N)$ and $Q = (L_2 :_R N)$. Thus, $P = \sqrt{(L_1 :_R N)}$ and $Q = \sqrt{(L_2 :_R N)}$. Hence, either $P \subseteq Q$ or $Q \subseteq P$ and this completes the proof.

(b) The proof is similar to that of part (a). \square

In [17, 2.10], it is shown that, if R be a Noetherian ring, M a finitely generated multiplication R -module, N a proper submodule of M such that $\text{Ass}_R(M/N)$ is a totally ordered set, and $(N :_R M)$ is a 2-absorbing ideal of R , then N is a 2-absorbing submodule of M . In the following theorem we see that some of this conditions are redundant.

Theorem 3.9. *Let N be a submodule of a multiplication R -module M such that $(N :_R M)$ is a 2-absorbing ideal of R . Then N is a 2-absorbing submodule of M .*

Proof. As $(N :_R M) \neq R$, $N \neq M$. Let $a, b \in R$, $m \in M$, and $abm \in N$. Since M is a multiplication R -module, there exists an ideal I of R such that $Rm = IM$. Thus $abIM \subseteq N$. Hence, $abI \subseteq (N :_R M)$. Now by assumption, $ab \in (N :_R M)$ or $aI \subseteq (N :_R M)$ or $bI \subseteq (N :_R M)$. Therefore, $ab \in (N :_R M)$ or $aIM \subseteq N$ or $bIM \subseteq N$. Thus $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. \square

An R -module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$, equivalently, for each submodule N of M , we have $N = (0 :_M \text{Ann}_R(N))$ [7].

Theorem 3.10. *Let N be a submodule of a comultiplication R -module M . Then we have the following.*

- (a) *If $\text{Ann}_R(N)$ is a 2-absorbing ideal of R , then N is a strongly 2-absorbing second submodule of M . In particular, N is a 2-absorbing second submodule of M .*

- (b) If M is a cocyclic module and N is a 2-absorbing second submodule of M , then N is a strongly 2-absorbing second submodule of M .

Proof. (a) Let $a, b \in R$, K be a submodule of M , and $abN \subseteq K$. Then we have $\text{Ann}_R(K)abN = 0$. So by assumption, $\text{Ann}_R(K)aN = 0$ or $\text{Ann}_R(K)bN = 0$ or $abN = 0$. If $abN = 0$, we are done. If $\text{Ann}_R(K)aN = 0$ or $\text{Ann}_R(K)bN = 0$, then $\text{Ann}_R(K) \subseteq \text{Ann}_R(aN)$ or $\text{Ann}_R(K) \subseteq \text{Ann}_R(bN)$. Hence, $aN \subseteq K$ or $bN \subseteq K$ since M is a comultiplication R -module.

(b) By Proposition 2.9, $\text{Ann}_R(N)$ is a 2-absorbing ideal of R . Thus the result follows from part (a). \square

The following example shows that Theorem 3.10 (a) is not satisfied in general.

Example 3.11. By [7, 3.9], the \mathbb{Z} -module \mathbb{Z} is not a comultiplication \mathbb{Z} -module. The submodule $N = p\mathbb{Z}$ of \mathbb{Z} , where p is a prime number, is not strongly 2-absorbing second submodule. But $\text{Ann}_{\mathbb{Z}}(p\mathbb{Z}) = 0$ is a 2-absorbing ideal of R .

For a submodule N of an R -module M the *second radical* (or *second socle*) of N is defined as the sum of all second submodules of M contained in N and it is denoted by $\text{sec}(N)$ (or $\text{soc}(N)$). In case N does not contain any second submodule, the second radical of N is defined to be (0) (see [11] and [2]).

Theorem 3.12. Let M be a finitely generated comultiplication R -module. If N is a strongly 2-absorbing second submodule of M , then $\text{sec}(N)$ is a strongly 2-absorbing second submodule of M .

Proof. Let N be a strongly 2-absorbing second submodule of M . By Proposition 3.5 (a), $\text{Ann}_R(N)$ is a 2-absorbing ideal of R . Thus by [9, 2.1], $\sqrt{\text{Ann}_R(N)}$ is a 2-absorbing ideal of R . By [5, 2.12], $\text{Ann}_R(\text{sec}(N)) = \sqrt{\text{Ann}_R(N)}$. Therefore, $\text{Ann}_R(\text{sec}(N))$ is a 2-absorbing ideal of R . Now the result follows from Theorem 3.10 (a). \square

Lemma 3.13. Let $f : M \rightarrow \hat{M}$ be a monomorphism of R -modules. If \hat{L} is a completely irreducible submodule of $f(M)$, then $f^{-1}(\hat{L})$ is a completely irreducible submodule of M .

Proof. This is straightforward. \square

Lemma 3.14. Let $f : M \rightarrow \hat{M}$ be a monomorphism of R -modules. If L is a completely irreducible submodule of M , then $f(L)$ is a completely irreducible submodule of $f(M)$.

Proof. Let $\{\hat{N}_i\}_{i \in I}$ be a family of submodules of $f(M)$ such that $f(L) = \cap_{i \in I} \hat{N}_i$. Then $L = f^{-1}f(L) = f^{-1}(\cap_{i \in I} \hat{N}_i) = \cap_{i \in I} f^{-1}(\hat{N}_i)$. This implies that there exists $i \in I$ such that $L = f^{-1}(\hat{N}_i)$ since L is a completely irreducible submodule of M . Therefore, $f(L) = ff^{-1}(\hat{N}_i) = f(M) \cap \hat{N}_i = \hat{N}_i$, as requested. \square

Theorem 3.15. Let $f : M \rightarrow \hat{M}$ be a monomorphism of R -modules. Then we have the following.

- (a) If N is a strongly 2-absorbing second submodule of M , then $f(N)$ is a 2-absorbing second submodule of \hat{M} .
- (b) If N is a 2-absorbing second submodule of M , then $f(N)$ is a 2-absorbing second submodule of $f(M)$.

- (c) If \dot{N} is a strongly 2-absorbing second submodule of \dot{M} and $\dot{N} \subseteq f(M)$, then $f^{-1}(\dot{N})$ is a 2-absorbing second submodule of M .
- (d) If \dot{N} is a 2-absorbing second submodule of $f(M)$, then $f^{-1}(\dot{N})$ is a 2-absorbing second submodule of M .

Proof. (a) Since $N \neq 0$ and f is a monomorphism, we have $f(N) \neq 0$. Let $a, b \in R$, \dot{L} be a completely irreducible submodule of \dot{M} , and $abf(N) \subseteq \dot{L}$. Then $abN \subseteq f^{-1}(\dot{L})$. As N is strongly 2-absorbing second submodule, $aN \subseteq f^{-1}(\dot{L})$ or $bN \subseteq f^{-1}(\dot{L})$ or $abN = 0$. Therefore,

$$af(N) \subseteq f(f^{-1}(\dot{L})) = f(M) \cap \dot{L} \subseteq \dot{L}$$

or

$$bf(N) \subseteq f(f^{-1}(\dot{L})) = f(M) \cap \dot{L} \subseteq \dot{L}$$

or $abf(N) = 0$, as needed.

(b) This is similar to the part (a).

(c) If $f^{-1}(\dot{N}) = 0$, then $f(M) \cap \dot{N} = f(f^{-1}(\dot{N})) = f(0) = 0$. Thus $\dot{N} = 0$, a contradiction. Therefore, $f^{-1}(\dot{N}) \neq 0$. Now let $a, b \in R$, L be a completely irreducible submodule of M , and $abf^{-1}(\dot{N}) \subseteq L$. Then

$$ab\dot{N} = ab(f(M) \cap \dot{N}) = abff^{-1}(\dot{N}) \subseteq f(L).$$

As \dot{N} is strongly 2-absorbing second submodule, $a\dot{N} \subseteq f(L)$ or $b\dot{N} \subseteq f(L)$ or $ab\dot{N} = 0$. Hence $af^{-1}(\dot{N}) \subseteq f^{-1}f(L) = L$ or $bf^{-1}(\dot{N}) \subseteq f^{-1}f(L) = L$ or $abf^{-1}(\dot{N}) = 0$, as desired.

(d) By using Lemma 3.14, this is similar to the part (c). \square

Corollary 3.16. Let M be an R -module and $N \subseteq K$ be two submodules of M . Then we have the following.

- (a) If N is a strongly 2-absorbing second submodule of K , then N is a 2-absorbing second submodule of M .
- (b) If N is a strongly 2-absorbing second submodule of M , then N is a 2-absorbing second submodule of K .

Proof. This follows from Theorem 3.15 by using the natural monomorphism $K \rightarrow M$. \square

A non-zero submodule N of an R -module M is said to be a *weakly second submodule* of M if $rsN \subseteq K$, where $r, s \in R$ and K is a submodule of M , implies either $rN \subseteq K$ or $sN \subseteq K$ [4].

Proposition 3.17. Let N be a non-zero submodule of an R -module M . Then N is a weakly second submodule of M if and only if N is a strongly 2-absorbing second submodule of M and $\text{Ann}_R(N)$ is a prime ideal of R .

Proof. Clearly, if N is a weakly second submodule of M , then N is a strongly 2-absorbing second submodule of M and by [4, 3.3], $\text{Ann}_R(N)$ is a prime ideal of R . For the converse, let $abN \subseteq H$ for some $a, b \in R$ and submodule K of M such that neither $aN \subseteq H$ nor $bN \subseteq H$. Then $ab \in \text{Ann}_R(N)$ and so either $a \in \text{Ann}_R(N)$ or $b \in \text{Ann}_R(N)$. This contradiction shows that N is weakly second. \square

The following example shows that the two concepts of strongly 2-absorbing second submodule and weakly second submodule are different in general.

Example 3.18. Let p, q be two prime numbers, $N = \langle 1/p + \mathbb{Z} \rangle$, and $K = \langle 1/q + \mathbb{Z} \rangle$. Then $N \oplus K$ is not a weakly second submodule of the \mathbb{Z} -module $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{q^\infty}$. But $N \oplus K$ is a strongly 2-absorbing second submodule of the \mathbb{Z} -module $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{q^\infty}$.

Corollary 3.19. Let N be a submodule of an R -module M . Then

$$\begin{aligned} N \text{ is second} &\Rightarrow N \text{ is weakly second} \Rightarrow N \text{ is strongly 2-absorbing second} \\ &\Rightarrow N \text{ is 2-absorbing second.} \end{aligned}$$

In general, none of the above implications is reversible.

Proof. The first assertion follows from [4, 3.2], Proposition 3.17, and Example 3.2. The second assertion follows from [4, 3.2], Example 3.18, and Example 3.2. \square

Proposition 3.20. Let M be an R -module and $\{K_i\}_{i \in I}$ be a chain of strongly 2-absorbing second submodules of M . Then $\cup_{i \in I} K_i$ is a strongly 2-absorbing second submodule of M .

Proof. Let $a, b \in R$, H be a submodule of M , and $ab(\cup_{i \in I} K_i) \subseteq H$. Assume that $a(\cup_{i \in I} K_i) \not\subseteq H$ and $b(\cup_{i \in I} K_i) \not\subseteq H$. Then there are $m, n \in I$, where $aK_n \not\subseteq H$ and $bK_m \not\subseteq H$. Hence, for every $K_n \subseteq K_s$ and $K_m \subseteq K_d$, we have that $aK_s \not\subseteq H$ and $bK_d \not\subseteq H$. Therefore, for each submodule K_h such that $K_n \subseteq K_h$ and $K_m \subseteq K_h$ we have $abK_h = 0$. Hence $ab(\cup_{i \in I} K_i) = 0$, as needed. \square

Definition 3.21. We say that a 2-absorbing second submodule N of an R -module M is a *maximal strongly 2-absorbing second submodule* of a submodule K of M , if $N \subseteq K$ and there does not exist a strongly 2-absorbing second submodule H of M such that $N \subset H \subset K$.

Lemma 3.22. Let M be an R -module. Then every strongly 2-absorbing second submodule of M is contained in a maximal strongly 2-absorbing second submodule of M .

Proof. This is proved easily by using Zorn's Lemma and Proposition 3.20. \square

Definition 3.23. Let N be a submodule of an R -module M . We define the *strongly 2-absorbing second radical* of N as the sum of all strongly 2-absorbing second submodules of M contained in N and we denote it by $s.2.sec(N)$. In case N does not contain any strongly 2-absorbing second submodule, the strongly 2-absorbing second radical of N is defined to be (0) . We say that $N \neq 0$ is a *strongly 2-absorbing second radical submodule* of M if $s.2.sec(N) = N$.

Proposition 3.24. Let N and K be two submodules of an R -module M . Then we have the following.

- (a) If $N \subseteq K$, then $s.2.sec(N) \subseteq s.2.sec(K)$.
- (b) $s.2.sec(N) \subseteq N$.
- (c) $s.2.sec(s.2.sec(N)) = s.2.sec(N)$.
- (d) $s.2.sec(N) + s.2.sec(K) \subseteq s.2.sec(N + K)$.
- (e) $s.2.sec(N \cap K) = s.2.sec(s.2.sec(N) \cap s.2.sec(K))$.
- (g) If $N + K = s.2.sec(N) + s.2.sec(K)$, then $s.2.sec(N + K) = N + K$.

Proof. These are straightforward. \square

Corollary 3.25. Let N be a submodule of an R -module M . If $s.2.sec(N) \neq 0$, then $s.2.sec(N)$ is a strongly 2-absorbing second radical submodule of M .

Proof. This follows from Proposition 3.24 (c). \square

Theorem 3.26. Let M be an R -module. If M satisfies the descending chain condition on strongly 2-absorbing second radical submodules, then every non-zero submodule of M has only a finite number of maximal strongly 2-absorbing second submodules.

Proof. Suppose that there exists a non-zero submodule N of M such that it has an infinite number of maximal strongly 2-absorbing second submodules. Then $s.2.sec(N)$ is a strongly 2-absorbing second radical submodule of M and $s.2.sec(N)$ has an infinite number of maximal strongly 2-absorbing second submodules. Let S be a strongly 2-absorbing second radical submodule of M chosen minimal such that S has an infinite number of maximal strongly 2-absorbing second submodules. Then S is not strongly 2-absorbing second. Thus there exist $r, t \in R$ and a submodule L of M such that $rtS \subseteq L$ but $rS \not\subseteq L$, $tS \not\subseteq L$, and $rtS \neq 0$. Let V be a maximal strongly 2-absorbing second submodule of M contained in S . Then $V \subseteq (L :_S r)$ or $V \subseteq (L :_S t)$ or $V \subseteq (0 :_S rt)$ so that $V \subseteq s.2.sec((K :_S r))$ or $V \subseteq s.2.sec((K :_S t))$ or $V \subseteq s.2.sec((0 :_S rt))$. By the choice of S , the modules $s.2.sec((K :_S r))$, $s.2.sec((K :_S t))$, and $s.2.sec((0 :_S rt))$ have only finitely many maximal strongly 2-absorbing second submodules. Therefore, there is only a finite number of possibilities for the module S , which is a contradiction. \square

Corollary 3.27. Every Artinian R -module has only a finite number of maximal strongly 2-absorbing second submodules.

Theorem 3.28. Let M be an R -module. If E is an injective R -module and N is a 2-absorbing submodule of M such that $Hom_R(M/N, E) \neq 0$, then $Hom_R(M/N, E)$ is a strongly 2-absorbing second R -module.

Proof. Let $r, s \in R$. Since N is a 2-absorbing submodule of M , we can assume that $(N :_M rs) = (N :_M r)$ or $(N :_M rs) = M$. Since E is an injective R -module, by replacing M with M/N in [4, 3.13 (a)], we have $Hom_R(M/(N :_M r), E) = rHom_R(M/N, E)$. Therefore,

$$\begin{aligned} rsHom_R(M/N, E) &= Hom_R(M/(N :_M rs), E) = \\ &Hom_R(M/(N :_M r), E) = rHom_R(M/N, E) \end{aligned}$$

or

$$\begin{aligned} rsHom_R(M/N, E) &= Hom_R(M/(N :_M rs), E) = \\ &Hom_R(M/M, E) = 0, \end{aligned}$$

as needed \square

Theorem 3.29. Let M be a strongly 2-absorbing second R -module and F be a right exact linear covariant functor over the category of R -modules. Then $F(M)$ is a strongly 2-absorbing second R -module if $F(M) \neq 0$.

Proof. This follows from [4, 3.14] and Theorem 3.3 (c) \Leftrightarrow (d). \square

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